ON THE UNIVERSAL TILING CONJECTURE IN DIMENSION ONE

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ABSTRACT. We show that the spectral-tile implication in the Fuglede conjecture in dimension 1 is equivalent to a Universal Tiling Conjecture and also to similar forms of the same implication for some simpler sets, such as unions of intervals with rational or integer endpoints.

CONTENTS

1. INTRODUCTION

Definition 1.1. For $\lambda \in \mathbb{R}$ we denote by

$$e_\lambda(x) := e^{2\pi i \lambda x}, \quad (x \in \mathbb{R})$$

Let Ω be Lebesgue measurable subset of $\mathbb{R}$ with finite Lebesgue measure. We say that Ω is *spectral* if there exists a subset Λ of $\mathbb{R}$ such that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(\Omega)$. In this case Λ is called a *spectrum* for Ω.

We say that Ω *tiles $\mathbb{R}$ by translations* if there exists a subset $\mathcal{T}$ of $\mathbb{R}$ such that $\{\Omega + t : t \in \mathcal{T}\}$ is a partition of $\mathbb{R}$, up to Lebesgue measure zero.

Fuglede's conjecture was stated for arbitrary finite dimension in [Fug74]. It asserts that the tiling and the spectral properties are equivalent. Tao [Tao04] disproved one direction in the Fuglede conjecture in dimensions 5 or higher: there exists a union of cubes which is spectral but does not tile. Later, Tao's counterexample was improved to disprove both directions in Fuglede's conjecture for dimensions 3 or higher [KM06, FMM06]. In the cases of dimensions 1 and 2, both directions are still open.

We state here Fuglede's conjecture in dimension 1:

Conjecture 1.2. [Fug74] *A subset Ω of $\mathbb{R}$ of finite Lebesgue measure is spectral if and only if it tiles $\mathbb{R}$ by translations.*

We focus on the spectral-tile implication in the Fuglede conjecture and present some equivalent statements. One of the main ingredients that we will use is the fact that any spectrum is *periodic* (see [BM11, IK12]).

There are good reasons for our focus on the cases of the conjectures in one dimension. One reason is periodicity (see the next definition): it is known, in 1D, that the possible sets Λ serving as candidates for spectra, in the sense of Fuglede's conjecture (Conjecture 1.2), must be periodic. A second reason lies in the difference, from 1D to 2D, in the possibilities for geometric configurations of translation sets.

The Universal Tiling Conjecture (Conjecture 1.6) suggests a reduction of the implication from spectrum to tile in Fuglede's conjecture, to a consideration of finite subsets of $\mathbb{Z}$. Hence computations for the problems in 1D are arithmetic in nature, as opposed to geometric; and connections to classical Fourier series may therefore be more direct in 1D.

Definition 1.3. If Ω is spectral then any spectrum Λ is *periodic* with some period $p \neq 0$, i.e., $\Lambda + p = \Lambda$, and p is an integer multiple of $\frac{1}{|\Omega|}$. We call p a period for Λ. If $p = \frac{k(p)}{|\Omega|}$ with $k(p) \in \mathbb{N}$, then Λ has the form

$$(1.1) \qquad\qquad \Lambda = \{\lambda_0, \ldots, \lambda_{k(p)-1}\} + p\mathbb{Z},$$

with $\lambda_0, \ldots, \lambda_{k(p)-1} \in [0, p)$, see [BM11, IK12]. The reason that there are $k(p)$ elements of Λ in the interval $[0, p)$ can be seen also from the fact that the Beurling density of a spectrum Λ has to be $|\Omega|$, see [Lan67].

Remark 1.4. The assertions in Definition 1.3 follow from [IK12]. According to [IK12], if Ω has Lebesgue measure 1 and is spectral with spectrum Λ, with $0 \in \Lambda$, then Λ is periodic, the period p is an integer and Λ has the form

$$(1.2) \qquad\qquad \Lambda = \{\lambda_0 = 0, \lambda_1, \ldots, \lambda_{p-1}\} + p\mathbb{Z},$$

where λ_i in $[0, p)$ are some distinct real numbers.

Definition 1.5. Let A be a finite subset of $\mathbb{R}$. We say that A is *spectral* if there exists a finite set Λ in $\mathbb{R}$ such that $\{e_\lambda : \lambda \in \Lambda\}$ is a Hilbert space for $L^2(\delta_A)$ where δ_A is the atomic measure $\delta_A := \sum_{a \in A} \delta_a$ and δ_a is the Dirac measure at a. We call Λ a *spectrum* for A.

We formulate the following "Universal Tiling Conjecture" for a fixed number $p \in \mathbb{N}$:

Conjecture 1.6. [UTC(p)] *Let $p \in \mathbb{N}$. Let $\Gamma := \{\lambda_0 = 0, \lambda_1, \ldots, \lambda_{p-1}\}$ be a subset of $\mathbb{R}$ with p elements. Let $K > 0$. Assume Γ has a spectrum of the form $\frac{1}{p}A$ with $A \subset \mathbb{Z}$, $\max|A| \leq K$. Then there exists a subset $\mathcal{T}$ of $\mathbb{Z}$ such that for any spectrum of Γ of the form $\frac{1}{p}A'$ with $A' \subset \mathbb{Z}$, $\max|A'| \leq K$, the set A' tiles $\mathbb{Z}$ by $\mathcal{T}$.*

One of the main results in this paper shows the equivalence of the spectral-tile implication in Fuglede's conjecture and the Universal Tiling Conjecture.

Theorem 1.7. *The following affirmations are equivalent.*

(i) *The Universal Tiling Conjecture is true for all $p \in \mathbb{N}$.*

(ii) *Every bounded Borel spectral set tiles by translations.*

Moreover, if these statements are true and if Ω, $|\Omega| = 1$, is a bounded Borel set which has a spectrum with period p, then Ω tiles by a subset $\mathcal{T}$ of $\frac{1}{p}\mathbb{Z}$.

The term "universal tiling" appears also in [PW01] for a special class of tiles of $\mathbb{R}$, namely those that tile $\mathbb{R}^+$. On the dual side, the Universal Spectral Conjecture was introduced in [LW97] where it is proved that some sets Ω which tile by some special tiling set $\mathcal{T}$ have a spectrum Λ which depends only on $\mathcal{T}$. In [FMM06] it is proved that the Universal Spectral Conjecture is equivalent to the tile-spectral implication in Fuglede's conjecture, in the case of finite abelian groups. It is also remarked in [FMM06] that a Universal Tiling Conjecture is equivalent to the tile-spectral implication, again in the setting of finite abelian groups. Our Theorem 1.7 refers to the original setting in the Fuglede conjecture, in dimension 1.

The second main result in this paper shows that the spectral-tile implication in the Fuglede conjecture is equivalent to some formulations of this implication for some special classes of sets Ω: unions of intervals with rational or integer endpoints.

Theorem 1.8. *The following affirmations are equivalent:*

(i) *For every finite union of intervals with rational endpoints $\Omega = \cup_{i=1}^{n}(\alpha_i, \beta_i)$ with $|\Omega| = 1$, if Ω has a spectrum Λ with period p, then Ω tiles $\mathbb{R}$ by a subset $\mathcal{T}$ of $\frac{1}{p}\mathbb{Z}$.*

(ii) *For every finite union of intervals with integer endpoints $\Omega = \cup_{i=1}^{n}(\alpha_i, \beta_i)$, $|\Omega| = N$, if Ω has a spectrum Λ with minimal period $\frac{r}{N}$, $r \in \mathbb{Z}$, then $\frac{N}{r}$ is an integer (see Corollary 2.2) and Ω tiles $\mathbb{R}$ with a subset $\mathcal{T}$ of $\frac{N}{r}\mathbb{Z}$.*

(iii) *Every bounded Borel spectral set tiles by translations.*

2. Analysis of spectral sets

We begin with a few lemmas and propositions that exploit the periodicity of the spectrum to give some information about the structure of Ω.

Proposition 2.1. *If $\Omega = \cup_{i=1}^{n}(\alpha_i, \beta_i)$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n$ is spectral, $p \in \mathbb{N}$ and $\alpha_i, \beta_i \in \frac{1}{p}\mathbb{Z}$ for all $i = 1, \ldots, n$, then any spectrum Λ for Ω has p as a period.*

Proof. Let Λ be a spectrum for Ω. Take $\lambda \in \Lambda$ and assume $\lambda + p \notin \Lambda$. We have, for $\lambda' \in \Lambda$, $\lambda' \neq \lambda$:

$$\langle e_{\lambda+p}, e_{\lambda'} \rangle_{L^2(\Omega)} = \sum_{i=1}^{n} \frac{1}{2\pi i(\lambda + p - \lambda')}(e^{2\pi i(\lambda+p-\lambda')\beta_i} - e^{2\pi i(\lambda+p-\lambda')\alpha_i})$$

$$= \frac{1}{2\pi i(\lambda + p - \lambda')} \sum_{i=1}^{n}(e^{2\pi i(\lambda-\lambda')\beta_i} - e^{2\pi i(\lambda-\lambda')\alpha_i}) = \frac{\lambda - \lambda'}{\lambda + p - \lambda'} \langle e_\lambda, e_{\lambda'} \rangle_{L^2(\Omega)} = 0$$

Similarly $\langle e_{\lambda+p}, e_\lambda \rangle_{L^2(\Omega)} = 0$. But this would contradict the completeness of $\{e_\lambda : \lambda \in \Lambda\}$. $\square$

Corollary 2.2. *If $\Omega = \cup_{i=1}^{n}(\alpha_i, \beta_i)$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n$ is spectral and $\alpha_i, \beta_i \in \mathbb{Z}$ for all $i = 1, \ldots, n$ and if a spectrum Λ has minimal period $\frac{k}{|\Omega|}$ then k divides $|\Omega|$.*

Proof. $|\Omega|$ is an integer N. Then $\frac{1}{N}\Omega$ has measure 1, endpoints in $\frac{1}{N}\mathbb{Z}$ and spectrum $N\Lambda$ with period k. By Proposition 2.1, k has to divide N. $\qquad\square$

The next proposition can be found in e.g. [BM11, LW97, Ped96, IK12]. We include the proof for the benefit of the reader.

Proposition 2.3. *Let Ω be a bounded Borel set of measure 1. Assume that Ω is spectral with spectrum Λ, $0 \in \Lambda$, which has period p. Then Ω is a p-tile of $\mathbb{R}$ by $\frac{1}{p}\mathbb{Z}$-translations, i.e., for almost every $x \in \mathbb{R}$, there exist exactly p integers $j_1, \ldots, j_p$ such that x is in $\Omega + \frac{j_i}{p}$, $i = 1, \ldots, p$.*

Also, for a.e. x in Ω, there are exactly k integers $j_1, \ldots, j_p$ such that $x + \frac{j_i}{p}$ is in Ω for all $i = 1, \ldots, p$.

Proof. The statements follow if we prove that

$$(2.1) \qquad \frac{1}{p}\sum_{j \in \mathbb{Z}} \chi_\Omega\left(x + \frac{j}{p}\right) = 1 \text{ a.e. on } [0, \tfrac{1}{p}].$$

We can assume Λ contains 0 so $p\mathbb{Z} \subset \Lambda$. We compute the Fourier coefficients: for $l \in \mathbb{Z}$,

$$p\int_0^{\frac{1}{p}} \frac{1}{p}\sum_{j \in \mathbb{Z}} \chi_\Omega\left(x + \frac{j}{p}\right) e_{-lp}(x)\,dx = \sum_{j \in \mathbb{Z}} \int_{\frac{j}{p}}^{\frac{j+1}{p}} \chi_\Omega(x) e_{-lp}(x)\,dx = \int_{\mathbb{R}} \chi_\Omega(x) e_{-lp}(x)\,dx = \delta_l.$$

This implies (2.1).

$\qquad\square$

Definition 2.4. *For $\varphi \in L^\infty(\mathbb{R})$, define the multiplication operator $M(\varphi)$ on $L^2(\mathbb{R})$ by $M(\varphi)f = \varphi f$, $f \in L^2(\Omega)$.*

Lemma 2.5. *Let Ω be a bounded Borel set of measure 1. Assume that Ω is spectral with spectrum Λ, $0 \in \Lambda$ which has period p. Let $P(p\mathbb{Z})$ be the orthogonal projection in $L^2(\Omega)$ onto the closed subspace spanned by $\{e_{kp} : k \in \mathbb{Z}\}$. Then, for $f \in L^2(\Omega)$,*

$$(2.2) \qquad (P(p\mathbb{Z})f)(x) = \frac{1}{p}\sum_{j \in \mathbb{Z}} f\left(x + \frac{j}{p}\right), \text{ for a.e. } x \in \Omega.$$

(We define f to be zero outside Ω)

Proof. Note first that the function on the right side of (2.2) has period $\frac{1}{p}$ and therefore it is in the L^2-span of the functions e_{pl}, $l \in \mathbb{Z}$. We have, for $f \in L^2(\Omega)$, $l \in \mathbb{Z}$:

$$\int_\Omega \frac{1}{p}\sum_{j \in \mathbb{Z}} f\left(x + \frac{j}{p}\right) e_{-lp}(x)\,dx = \frac{1}{p}\sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} \chi_\Omega\left(x - \frac{j}{p}\right) f(x) e_{-lp}(x)\,dx$$

$$= \text{(with Proposition 2.3)} = \int_\Omega f(x) e_{-lp}(x)\,dx = \langle f, e_{lp}\rangle_{L^2(\Omega)}.$$

Therefore, since e_{lp} form an orthogonal basis for their span, equation (2.2) follows.

$\qquad\square$

Proposition 2.6. *Let Ω be a bounded Borel set of measure 1. Assume that Ω is spectral with spectrum Λ, which has period p and assume $0 \in \Lambda$. Let $\Lambda = \{\lambda_0 = 0, \lambda_1, \ldots, \lambda_{p-1}\} + p\mathbb{Z}$ with $\lambda_i \in [0, p)$, $i = 0, \ldots, p-1$. Then the projection $P(\lambda_i + p\mathbb{Z})$ onto the span of $\{e_{\lambda_i + kp} : k \in \mathbb{Z}\}$ has the following formula: for $f \in L^2(\Omega)$,*

$$(2.3) \qquad (P(\lambda_i + p\mathbb{Z})f)(x) = e_{\lambda_i}(x)\frac{1}{p}\sum_{j \in \mathbb{Z}} f\left(x + \frac{j}{p}\right)e_{-\lambda_i}\left(x + \frac{j}{p}\right), \text{ for a.e. } x \in \Omega.$$

Proof. A simple check shows that for $\lambda \in \Lambda$ we have that the rank one projections $P(\lambda)$ onto e_λ are related by the following formula:

$$(2.4) \qquad\qquad\qquad P(\lambda) = M(e_\lambda)P(0)M(e_{-\lambda}).$$

Then, we obtain

$$(2.5) \qquad\qquad\qquad P(\lambda_i + p\mathbb{Z}) = M(e_{\lambda_i})P(p\mathbb{Z})M(e_{-\lambda_i}).$$

Equation (2.3) follows from (2.5) and (2.2). $\qquad\square$

Proposition 2.7. *Let $\Lambda = \{\lambda_0 = 0, \lambda_1, \ldots, \lambda_{k(p)-1}\} + p\mathbb{Z}$ be as in Proposition 2.6. For $x \in \mathbb{R}$, let $\Omega_x = \{j \in \mathbb{Z} : x + \frac{j}{p} \in \Omega\}$. Then $|\Omega_x| = p$ for a.e. $x \in \mathbb{R}$ and, for $i, i' = 0, \ldots, p-1$:*

$$(2.6) \qquad\qquad\qquad \frac{1}{p}\sum_{j \in \Omega_x} e^{2\pi i(\lambda_i - \lambda_{i'})\frac{j}{p}} = \delta_{ii'} \text{ for a.e. } x \in \Omega.$$

In other words , the set $\{\lambda_0, \ldots, \lambda_{p-1}\}$ is spectral and, for a.e. $x \in \mathbb{R}$, $\frac{1}{p}\Omega_x$ is a spectrum for it.

Proof. The first statement is contained in Proposition 2.3. Equation (2.6) follows from Proposition 2.6 and the fact that $P(\lambda_i + p\mathbb{Z})e_{\lambda_{i'}} = 0$ for $i \neq i'$, because Λ is a spectrum. $\qquad\square$

Proposition 2.8. *Let Ω be a bounded Borel set of measure 1. Assume that Ω is spectral with spectrum Λ as in Proposition 2.7 and, for $x \in \Omega$ define $\Omega_x \subset \mathbb{Z}$, $0 \in \Omega_x$ as in Proposition 2.7. The sets Ω_x have a common spectrum $\frac{1}{p}\{\lambda_0 = 0, \lambda_1, \ldots, \lambda_{p-1}\}$ for a.e. $x \in \Omega$. Assume in addition that the sets Ω_x tile $\mathbb{Z}$ by a common tiling set $\mathcal{T} \subset \mathbb{Z}$. Then Ω tiles $\mathbb{R}$ by $\frac{1}{p}\mathcal{T}$.*

Proof. The fact that $\frac{1}{p}\Omega_x$ have the common spectrum $\{\lambda_i\}$ is contained in Proposition 2.7. We focus on the tiling property. We can assume $0 \in \mathcal{T}$. We know from Proposition 2.3 that Ω p-tiles $\mathbb{R}$ by $\frac{1}{p}\mathbb{Z}$. Take $y \in \mathbb{R}$. Then there exists $k \in \mathbb{Z}$ and $x \in \Omega$ such that $x + \frac{k}{p} = y$. Since Ω_x tiles $\mathbb{Z}$ with $\mathcal{T}$, there exist $j \in \Omega_x$ and $t \in \mathcal{T}$ such that $k = j + t$. Then $x + \frac{j}{p} \in \Omega$ and $x + \frac{j}{p} + \frac{t}{p} = y$. So $\cup_{t \in \mathcal{T}}(\Omega + \frac{t}{p})$ covers $\mathbb{R}$.

Now assume $y = x_1 + \frac{t_1}{p} = x_2 + \frac{t_2}{p}$ with $x_1, x_2 \in \Omega$ and $t_1, t_2 \in \mathcal{T}$. Then $x_1 - x_2 \in \frac{1}{p}\mathbb{Z}$ and therefore, there exists $j \in \Omega_{x_1}$ such that $x_2 = x_1 + \frac{j}{p}$. Also $0 \in \Omega_{x_1}$ since $x_1 \in \Omega$. Then $x_1 + \frac{j}{p} + \frac{t_2}{p} = x_1 + \frac{t_1}{p}$ so $\frac{j}{p} + \frac{t_2}{p} = 0 + \frac{t_1}{p}$. But since Ω_{x_1} tiles with $\mathcal{T}$, this implies $j = 0$ and $t_1 = t_2$, so $x_1 = x_2$. Thus the sets $\Omega + t$ are mutually disjoint. $\qquad\square$

Proposition 2.9. *Let Ω be a bounded Borel subset of $\mathbb{R}$ with $|\Omega| = 1$. Let $p \in \mathbb{N}$. Suppose Ω p-tiles $\mathbb{R}$ by $\frac{1}{p}\mathbb{Z}$. Then, for a.e. $x \in \mathbb{R}$ the set*

$$(2.7) \qquad \Omega_x := \left\{ k \in \mathbb{Z} : x + \frac{k}{p} \in \Omega \right\}$$

has exactly p elements

$$(2.8) \qquad \Omega_x = \{ k_0(x) < k_1(x) < \cdots < k_{p-1}(x) \}.$$

For almost every $x \in \mathbb{R}$ there exist unique $y \in [0, \frac{1}{p})$ and $i \in \{0, \ldots, p-1\}$ such that $y + \frac{k_i(y)}{p} = x$.

The functions k_i have the following property

$$(2.9) \qquad k_i(x + \frac{1}{p}) = k_i(x) - 1, \quad (x \in \mathbb{R}, i = 0, \ldots, p-1).$$

Consider the space of $\frac{1}{p}$-periodic vector valued functions $L^2([0, \frac{1}{p}), \mathbb{C}^p)$. The operator $W : L^2(\Omega) \to L^2([0, \frac{1}{p}), \mathbb{C}^p)$ defined by

$$(2.10) \qquad (Wf)(x) = \begin{pmatrix} f\left(x + \frac{k_0(x)}{p}\right) \\ \vdots \\ f\left(x + \frac{k_{p-1}(x)}{p}\right) \end{pmatrix}, \quad (x \in [0, \frac{1}{p}), f \in L^2(\Omega)),$$

is an isometric isomorphism with inverse

$$(2.11) \qquad W^{-1} \begin{pmatrix} f_0 \\ \vdots \\ f_{p-1} \end{pmatrix} (x) = f_i(y), \ \text{ if } x = y + \frac{k_i(y)}{p}, \ \text{ with } y \in [0, \frac{1}{p}), i \in \{0, \ldots, p-1\}.$$

A set Λ of the form $\Lambda = \{0 = \lambda_0, \lambda_1, \ldots, \lambda_{p-1}\} + p\mathbb{Z}$ is a spectrum for Ω if and only if $\{\lambda_0, \ldots, \lambda_{p-1}\}$ is a spectrum for $\frac{1}{p}\Omega_x$ for a.e. $x \in [0, \frac{1}{p})$.

The exponential functions are mapped by W as follows:

$$(2.12) \qquad (We_{\lambda_i + np})(x) = e_{\lambda_i + np}(x) \begin{pmatrix} e_{\lambda_i}(\frac{k_0(x)}{p}) \\ \vdots \\ e_{\lambda_i}(\frac{k_{p-1}(x)}{p}) \end{pmatrix} =: F_{i,n}(x), \quad (i = 0, \ldots, p-1, n \in \mathbb{Z}, x \in [0, \frac{1}{p})).$$

Proof. The first statement follows from the fact that Ω p-tiles $\mathbb{R}$ by $\frac{1}{p}\mathbb{Z}$. The second statement follows from this and the fact that $[0, \frac{1}{p})$ tiles $\mathbb{R}$ by $\frac{1}{p}\mathbb{Z}$.

To check that W and W^{-1}, as defined, are inverse to each other requires just a simple computation. We verify that W is isometric.

For a subset S of $\mathbb{Z}$ with $|S| = p$ define

$$A_S := \{ x \in [0, \frac{1}{p}) : \Omega_x = S \}.$$

Note that, since Ω is bounded, $A_S = \emptyset$ for all but finitely many sets S. Also we have the following partition of Ω.

$$\bigcup_{|S|=p} (A_S + \frac{1}{p}S) = \Omega.$$

Take $f \in L^2(\Omega)$. We have

$$\|Wf\|_{L^2([0,\frac{1}{p}),\mathbb{C}^p)}^2 = \int_0^{\frac{1}{p}} \sum_{j=0}^{p-1} \left| f\left(x + \frac{k_j(x)}{p}\right) \right|^2 dx = \sum_{|S|=p} \int_{A_S} \sum_{j=0}^{p-1} \left| f\left(x + \frac{k_j(x)}{p}\right) \right|^2 dx$$

$$= \sum_{|S|=p} \int_{A_S} \sum_{s \in S} \left| f\left(x + \frac{s}{p}\right) \right|^2 dx = \sum_{|S|=p} \sum_{s \in S} \int_{A_S + \frac{s}{p}} |f(x)|^2 dx = \int_\Omega |f(x)|^2 dx.$$

Equation (2.12) requires just a simple computation.

If Λ is a spectrum for Ω, then we saw in Proposition 2.7 that $\frac{1}{p}\Omega_x$ has spectrum $\{\lambda_0, \ldots, \lambda_{p-1}\}$ for a.e. $x \in \mathbb{R}$.

For the converse, if $\{\lambda_0, \ldots, \lambda_{p-1}\}$ is a spectrum for $\frac{1}{p}\Omega$ for a.e. $x \in [0, \frac{1}{p})$, then for a.e. $x \in [0, \frac{1}{p})$,the vectors $v_i(x) = \frac{1}{\sqrt{p}}(e_{\lambda_i}(\frac{k_0(x)}{p}), \ldots, e_{\lambda_i}(\frac{k_{p-1}(x)}{p}))^t$, $i = 0, \ldots, p-1$, form an orthonormal basis for $\mathbb{C}^p$. Then the functions $F_{i,n}$ in (2.12) form an orthonormal basis for $L^2([0, \frac{1}{p}), \mathbb{C}^p)$ as can be seen by a short computation. To see that the functions $F_{i,n}$ span the entire Hilbert space, take H in $L^2([0, \frac{1}{p}), \mathbb{C}^p)$ such that $H \perp F_{i,n}$ for all $i = 0, \ldots, p-1$, $n \in \mathbb{Z}$. Then

$$0 = \int_0^{\frac{1}{p}} e_{\lambda_i + np}(x) \langle H(x), v_i(x) \rangle_{\mathbb{C}^p} dx.$$

Since the functions e_{np} are complete in $L^2[0, \frac{1}{p})$ it follows that $e_{\lambda_i}(x) \langle H(x), v_i(x) \rangle_{\mathbb{C}^p} = 0$ for a.e. $x \in [0, \frac{1}{p})$. So $\langle H(x), v_i(x) \rangle = 0$ for a.e. $x \in [0, \frac{1}{p})$ and all $i = 0, \ldots, p-1$. Then $H(x) = 0$ for a.e. $x \in [0, \frac{1}{p})$.

$\square$

We can prove now our main results.

Proof of Theorem 1.7. Assume (i). Let Ω be a bounded Borel spectral set. By scaling we can assume $|\Omega| = 1$. Then Ω has a spectrum of period p. The sets Ω_x in Proposition 2.7 are obviously bounded by a common number K, since Ω is bounded. Then just apply the UTC(p) conjecture to the sets Ω_x and use Proposition 2.8. We get also that Ω tiles by a subset of $\frac{1}{p}\mathbb{Z}$.

For the converse, assume (ii) holds. Let $p \in \mathbb{N}$, $K > 0$. Take $\Gamma = \{\lambda_0 = 0, \ldots, \lambda_{p-1}\}$ which has a spectrum of the form $\frac{1}{p}A$ with $A \subset \mathbb{Z}$, $\max|A| \leq K$. Take all the possible sets $A_1, \ldots, A_n$ in $\mathbb{Z}$ such that $\frac{1}{p}A_i$ is a spectrum for Γ and $\max|A_i| \leq K$.

Now pick $0 = r_1 < r_2 < \cdots < r_n < r_{n+1} = \frac{1}{p}$ with the property that $r_i - r_j \notin \mathbb{Q}$ unless $i = j$ or $\{i, j\} = \{1, n+1\}$. Define the set Ω as follows:

$$\Omega := \bigcup_{i=1}^n \left([r_i, r_{i+1}) + \frac{1}{p}A_i \right).$$

For every $x \in [0, \frac{1}{p})$, the sets Ω_x defined in Proposition 2.7 are among the sets A_i. Using Proposition 2.9, since Γ is a spectrum for all the sets $\frac{1}{p}\Omega_x$, we get that Ω is spectral with spectrum $\Gamma + p\mathbb{Z}$. By hypothesis Ω tiles $\mathbb{R}$ with some set $\mathcal{T}$. We can assume that $0 \in \mathcal{T}$.

We prove first that $\mathcal{T}$ is contained in $\frac{1}{p}\mathbb{Z}$. Note that $|\Omega| = 1$. By [LW96, Theorem 1 and 2], we know that $\mathcal{T}$ is periodic with some integer period s and $\mathcal{T} = \{0 = t_0, t_1, \dots, t_J\} + s\mathbb{Z}$ with t_i rational for all i.

We claim that, if $\overline{(\Omega + t)} \cap \overline{(\Omega + t')} \neq \emptyset$ then $t - t' \in \frac{1}{p}\mathbb{Z}$. Indeed, if this is the case, from the tiling property, we see that either $t = t'$ or two intervals, one in $\Omega + t$, one in $\Omega + t'$ have a common endpoint. So we have $r_{i+1} + \frac{s_i}{p} + t = r_j + \frac{s_j}{p} + t'$ for some $i, j \in \{1, \dots, n\}$, $s_i \in A_i, s_j \in A_j$. But, since $t, t' \in \mathbb{Q}$, we get that $r_{i+1} - r_j$ is in $\mathbb{Q}$. So $i + 1 = j$ or $\{i + 1, j\} = \{1, n + 1\}$. In both cases $r_{i+1} - r_j \in \frac{1}{p}\mathbb{Z}$. Then $t - t' \in \frac{1}{p}\mathbb{Z}$.

Next, consider the set

$$\mathcal{R} := \left\{ x \in \mathbb{R} : \text{ There exist } t_1 = 0, \dots, t_N \in \mathcal{T} \text{ such that } \overline{(\Omega + t_i)} \cap \overline{(\Omega + t_{i+1})} \text{ for all } i, \text{ and } x \in \overline{(\Omega + t_N)} \right\}.$$

We claim that $\mathcal{R}$ is both open and closed. First, we check it is open: if $x \in \mathcal{R}$, then let $t_1, \dots, t_N$ as before, $x \in \overline{(\Omega + t_N)}$. If x is interior to $\Omega + t_N$ then the entire $\Omega + t_N$ is contained in $\mathcal{R}$. If x is a boundary point for $\Omega + t_N$, then it appears also as a boundary point for some other $\Omega + t_{N+1}$. The union $\overline{(\Omega + t_N)} \cup \overline{(\Omega + t_{N+1})}$ contains an open neighborhood for x and is contained in $\mathcal{R}$. To see that $\mathcal{R}$ is closed one can use a similar discussion. Clearly Ω is contained in $\mathcal{R}$ so $\mathcal{R}$ is non-empty. The fact that $\mathbb{R}$ is connected implies that $\mathcal{R} = \mathbb{R}$. But this implies that for every $t \in \mathcal{T}$, there exist $t_0 = 0, \dots, t_N = t$ such that $\overline{(\Omega + t_i)} \cap \overline{(\Omega + t_{i+1})} \neq \emptyset$ for all i. With the previous fact, we get that $t_{i+1} - t_i \in \frac{1}{p}\mathbb{Z}$ and, by induction $t \in \frac{1}{p}\mathbb{Z}$.

We prove now that $\frac{1}{p}A_i$ tiles $\frac{1}{p}\mathbb{Z}$ by $\mathcal{T}$, for all i, which proves (i).

Pick $y \in (r_i, r_{i+1})$ and $k \in \mathbb{Z}$. Then there exists a unique $t \in \mathcal{T}$ such that $y + \frac{k}{p} \in \Omega + t$. So there exists $1 \leq j \leq n$, $y' \in [r_j, r_{j+1})$ and $s_j \in A_j$ such that $y + \frac{k}{p} = y' + \frac{s_j}{p} + t$. This implies $y - y' \in \frac{1}{p}\mathbb{Z}$ which means that $y = y'$ and $i = j$. So $\frac{k}{p} = \frac{s_j}{p} + t$.

Now, assume $\frac{a}{p} + t = \frac{a'}{p} + t'$ for some $a, a' \in A_i$ and $t, t' \in \mathcal{T}$. Then

$$(\Omega + t) \cap (\Omega + t') \supset ([r_i, r_{i+1}) + \frac{a}{p} + t) \cap ([r_i, r_{i+1}) + \frac{a'}{p} + t') = [r_i, r_{i+1}) + \frac{a}{p} + t.$$

This implies that $t = t'$ and $a = a'$. So $\frac{1}{p}A_i$ tiles $\frac{1}{p}\mathbb{Z}$ by $\mathcal{T}$.

$\square$

Proof of Theorem 1.8. That (iii) implies (i) follows from Theorem 1.7. To see that (i) implies (iii), we show that the Universal Tiling Conjecture is true under these assumptions. The proof is similar to the one for Theorem 1.7. Let p, Γ, K as in Conjecture 1.6. Let $A_1, \dots, A_m$ be all the possible spectra for Γ which are contained in $\mathbb{Z}$ and with $\max |A_i| \leq K$.

Now pick $0 = r_1 < r_2 < \dots < r_{m+1} = \frac{1}{p}$ some rational points and define

$$\Omega = \bigcup_{i=1}^{m} \left((r_i, r_{i+1}) + \frac{1}{p}A_i \right).$$

For every $x \in [0, \frac{1}{p})$, the sets Ω_x defined in Proposition 2.7 are among the sets A_i. Using Proposition 2.9, since Γ is a spectrum for all the sets $\frac{1}{p}\Omega_x$, we get that Ω is spectral with spectrum $\Gamma + p\mathbb{Z}$. By hypothesis Ω tiles $\mathbb{R}$ with some set $\mathcal{T}$ contained in $\frac{1}{p}\mathbb{Z}$. We can assume $0 \in \mathcal{T}$. We prove that every A_i tiles $\mathbb{Z}$ by $p\mathcal{T}$.

Take $x \in (r_i, r_{i+1})$ and $k \in \mathbb{Z}$. Then there exists $t \in \mathcal{T}$ such that $x + \frac{k}{p} \in \Omega + t$, so there exist $j \in \{1, \dots, m\}$, $y \in (r_j, r_{j+1})$ and $s \in A_j$ such that $x + \frac{k}{p} = y + \frac{s}{p} + t$. This implies that $x - y \in \frac{1}{p}\mathbb{Z}$ and since $x, y \in [0, \frac{1}{p})$, this means that $x = y$. Hence $i = j$ and $k = s + pt$.

Now assume $s + pt = s' + pt'$ for some $s, s' \in A_i$ and $t, t' \in \mathcal{T}$. Then $(\Omega + t) \cap (\Omega + t')$ contains $(r_i, r_{i+1}) + \frac{s}{p} + t = (r_i, r_{i+1}) + \frac{s'}{p} + t'$. So $t = t'$ and $s = s'$. Thus, A_i tiles $\mathbb{Z}$ by $p\mathcal{T}$.

Assume (i) and take Ω as in (ii). Then $\frac{1}{N}\Omega$ has rational endpoints and measure 1. If Λ is a spectrum for Ω with period $\frac{r}{N}$, then $N\Lambda$ is a spectrum for $\frac{1}{N}\Omega$ with period r. By assumption, $\frac{1}{N}\Omega$ tiles $\mathbb{R}$ by a subset of $\frac{1}{r}\mathbb{Z}$. Then Ω tiles $\mathbb{R}$ by a subset $\frac{N}{r}\mathbb{Z}$.

Assume (ii) and take Ω as in (i). Let N be common denominator of all α_i, β_i. Then $N\Omega$ has integer endpoints and measure N. If Λ is a spectrum for Ω of period r, then $\frac{1}{N}\Lambda$ is a spectrum for $N\Omega$, with period $\frac{r}{N}$ (which is a multiple of the minimal period $\frac{r'}{N}$). By assumption, $N\Omega$ tiles $\mathbb{R}$ by a subset of $\frac{N}{r'}\mathbb{Z} \subset \frac{N}{r}\mathbb{Z}$. Then Ω tiles $\mathbb{R}$ by a subset of $\frac{1}{r}\mathbb{Z}$. $\qquad\square$

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